Inequality with side lengths, altitudes and circumradius.
https://www.linkedin.com/feed/update/urn:li:activity:6556410707775238144
In an arbitrary triangle ABC, let a, b, c denote the lengths of the sides, R its circumradius, and let h_a, h_b, h_c respectively, denote the lengths of the corresponding altitudes. Prove the inequality, and give the conditions under which equality holds.

$$\sum \frac{a^2+bc}{b+c} \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}}.$$

Solution by Arkady Alt, San Jose, California, USA.

Let F be area of $\triangle ABC$. Since $\frac{3abc}{2R}\left(\frac{1}{h_a h_b h_c}\right)^{1/3} = \frac{3 \cdot 4RF}{2R}\left(\frac{abc}{ah_a \cdot bh_b \cdot ch_c}\right)^{1/3} = 6F\left(\frac{abc}{8F^3}\right)^{1/3} = 3(abc)^{1/3}$ and by AM-GM Inequality $3(abc)^{1/3} \leq a+b+c$ remains to prove inequality $\sum \frac{a^2+bc}{b+c} \geq a+b+c \Leftrightarrow \sum\left(\frac{a^2+bc}{b+c} - a\right) \geq 0 \Leftrightarrow \sum \frac{(a-b)(a-c)}{b+c} \geq 0 \Leftrightarrow \sum(a^2-b^2)(a^2-c^2) \geq 0.$

We have $\sum(a^2-b^2)(a^2-c^2) = \sum(a^4 - a^2b^2 - a^2c^2 + b^2c^2) = \sum a^4 - \sum b^2c^2 = \frac{1}{2}\sum(a^4+b^4-2a^2b^2) = \frac{1}{2}\sum(a^2-b^2)^2 \geq 0.$

Since both inequalities which was used becomes equality iff $a=b=c$ then inequality of the problem becomes equality iff $\triangle ABC$ is equilateral. •